Nonlinear effect on homogenized constitutive laws of soils

Jian-Guo Wang Y. Ichikawa
(王 建国 市川 康明)
学生会員 正会員
名古屋大学工学部地圏環境工学科

[ABSTRACT] The nonlinear homogenized constitutive law is analyzed by CELL.FORT if strain path is given. Two simple cases have been shown. The results are as followings: 1) The distribution of stress level in a unit cell has vital effect on the mechanical properties of mixed soil. The effect of stress level is dependent on the volume fraction and distribution patterns of inclusions. 2) Stress level in the zone near the interface is changing rapidly to relax the gradient of micro-stress field. 3) Initial anisotropy will vanish with progress of the deformation. (4) The nonlinear effect is the results of microstructure evolution which is depending on the stress/strain path.

1. Introduction

There are many methods to improve soil properties. One is to make a mixed soil. In this case, hard particles are used to form skeleton, and soft clay forms matrix. The loads on this composite foundation are distributed among skeleton and matrix through its microstructures. It is very important to find an appropriate method to evaluate the reinforcement effect of foundation improvement. Until now there are several methods to study this problem but no one considers the nonlinear effect on the constitutive laws of composite soils, specially the interaction effect of nonlinear constituents in micro-scale.

There are many methods to estimate the overall properties of media with heterogeneous microstructures, which were proposed by such as Voigt, Reuss, Taylor[1]; Horii & Nemat-Nasser[2], Hori[3], Ortiz[18] etc. One of the most powerful methods for periodic microstructures is the homogenization theory, firstly used by Sanchez-Palencia (1970-1974) in seepage problem[4]. This method requires that the medium is periodic or well random[19]. It has been shown that structures of a unit cell do not affect the effective properties of strength noticeably and volume fraction is the most important quantity for linear elastic media special isotropic linear media[5,6]. Another characteristic of a linear problem is complete separation of microscopic description from macroscopic description. But this is not true for nonlinear materials specially when they are in the intense interaction among the constituents[6~11]. Homogenization theory is advantageous over others when the interaction is intensified during nonlinear zone of material properties[12,13]. Now an increasing tendency on its mathematical aspects of nonlinear homogenization is seen in both mathematical aspects and application aspects [6~17]. The main concerned point for mathematicians is the convergence of nonlinear homogenization[14,15], because the convergence of an asymptotic expansion is the most pressing problem for any type of application problems. Many methods are proposed to improve the convergence such as correctors [9,14] with two-scale Young measure. Homogenization has the advantage that it is a mathematical technique based on asymptotics. This fact allows one to proceed formally, and to apply it over a wide variety of physical problems (linear or nonlinear) described by partial differential equations. Jansson studied the effect of nonlinear constitutive equation with power-law on overall behaviors of fiber-reinforced materials and found that some components of linear elastic response is not greatly affected by the array type, but nonlinear one is quite different[6]. Auriault[13] used power type stress strain relationship (Ramberg-Osgood model) to describe hot compaction of metal powders and found that stress-strain relation remains the same form both in micro-scale level and in macro-level. Devries et al[16] introduced damage variables into homogenization of composite materials. Santosa and Symes[17] considered an internal dissipation boundary in a unit cell and discovered the boundary only appears in local problem and Paipetis et. al[22] extended to three-dimensional case. But in authors' knowledge few publications on application of homogenization theory to nonlinear properties of each constituent, specially for geomaterials, are published.

This paper extends the homogenization method to the case including nonlinear constitutive laws of each constituent and applies it to an imagined mixed soil. The mixed soils are regarded as composite materials with **periodic microstructures**$^+$, whose smallest repeatable element is called unit cell. The unit cell is composed of hard constituents such as stone or sand, the skeleton, and soft matrix such as soft clay. Here we just assume that the skeleton is linear elastic but the matrix is Duncan-Chang's nonlinear materials. Isotropic compression and Ko-compression cases are computed to study the constitutive properties of the mixture soils by homogenized theory for plane strain problem as beginnings.

+ '**Periodic microstructures**' is being evolved during loading, but defined at any time or time interval.

2. Formulation of the problem in an incremental form

Infinitesimal deformation process is discussed. There is a body occupying domain Ω and it is subjected to a system of body for $f(x,y;t)$ and surface force $F(x;t)$ on $\Gamma_F \times [0,T]$. Other part of its whole boundary Γ is fixed ($\Gamma = \Gamma_0 \cup \Gamma_F$). Its strong form in time interval $[t,\ t+\Delta t]$, denoting ΔT, can be regarded as parameterized equation set. It is a usual linear equation set parameterized t. Time t can express the history of loading or true time:

Governing equations

Equilibrium equation $\dfrac{\partial \Delta\sigma_{ij}^{\varepsilon}}{\partial x_j} + \Delta f_i^{\varepsilon}(x;t) = 0$ *in* $\Omega^{\varepsilon} \times \Delta T$ (1)

Geometrical relation $\Delta\varepsilon_{ij}(u^{\varepsilon}) = \dfrac{1}{2}\left(\dfrac{\partial \Delta u_i^{\varepsilon}}{\partial x_j} + \dfrac{\partial \Delta u_j^{\varepsilon}}{\partial x_i}\right)$ *in* $\Omega^{\varepsilon} \times \Delta T$ (2)

Constitutive equation $\Delta\sigma_{ij}^{\varepsilon} = E_{ijkl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}; t^*)\Delta\varepsilon_{kl}(\Delta u^{\varepsilon})$ *in* $\Omega^{\varepsilon} \times \Delta T$ (3)++

Boundary condition: $\Delta\sigma_{ij}^{\varepsilon} n_j = \Delta F_i(x;t)$ on $\Gamma_F \times \Delta T$ and $\Delta u_i^{\varepsilon} = \Delta\bar{u}_i(x;t)$ on $\Gamma_o \times \Delta T$.

where $\varepsilon = \dfrac{\mathbf{x}}{\mathbf{y}}$ denotes Y-periodicity and $\varepsilon << 1$. $\mathbf{y}$ is the fast spatial variable and $\mathbf{x}$ is the slow spatial variable. $E_{ijkl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}; t) = E_{jikl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}; t) = E_{ijlk}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}; t) = E_{jilk}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}; t)$ is the function of strain /stress history parametrized by time t, generally expressing a linear or nonlinear constitutive law. $u^{\varepsilon}(x,y;t) = u(x,y;t)|_{y=\frac{x}{\varepsilon}}$, $\sigma_{ij}^{\varepsilon} = \sigma_{ij}(x,y;t)|_{y=\frac{x}{\varepsilon}}$, $\varepsilon_{ij}^{\varepsilon} = \varepsilon_{ij}(x,y;t)|_{y=\frac{x}{\varepsilon}}$ are displacement vector, stress tensor and strain tensor respectively. They are all Y-periodicity at any time t.

++ Generalized constitutive law can be expressed as $\{d\sigma\} = [D]_{ep}\{d\varepsilon\}$. Its finite incremental form is $\{\Delta\sigma\} = \int_{\varepsilon}^{\varepsilon+\Delta\varepsilon} [D]_{ep}\{d\varepsilon\} = g(\Delta\varepsilon) = [E(x, \varepsilon_{rs}^{\varepsilon}; t^*)]\{\Delta\varepsilon\}$ and $t^* \in [t,\ t+\Delta t]$#

3. Asymptotic expansion

The homogenized constitutive law in incremental form[24]

$$< \Delta\sigma_{ij} >= E_{ijkl}^{h}(x, \varepsilon_{rs}^{0})\Delta\varepsilon_{kl}^{0} \quad (4)$$

where $E_{ijkl}^{h}(x,\varepsilon_{rs}^{0}) = \dfrac{1}{|Y|}\int_Y E_{qnpm}(x,\varepsilon_{rs}^{0})[\delta_{qi}\delta_{nj} - \dfrac{\partial W_q^{ij}}{\partial y_n}][\delta_{pk}\delta_{ml} - \dfrac{\partial W_p^{kl}}{\partial y_m}]dy$ (5)+++

and characteristic functions W_p^{kl} are determined by the following unit cell problem:

$$\Delta u_i^1(x,y;t) = -W_i^{kl}(y)\frac{\partial \Delta u_k^0}{\partial x_l} + c(x;t) \quad (6) \qquad \frac{\partial}{\partial y_l}\{E_{klij}^{\varepsilon}(x,\varepsilon_{rs}^{0})[\Delta\varepsilon_{ij}^{o} + \Delta\varepsilon_{ij}^{1}]\} = 0 \quad (7)$$

with Y-periodicity boundary condition, we solve

$\displaystyle\int_Y E_{klij}^{\varepsilon}(x,\varepsilon_{rs}^{0})\frac{\partial W_i^{pq}(y)}{\partial y_j}\frac{\partial V_k}{\partial y_l}dy = \int_Y E_{klpq}^{\varepsilon}(x,\varepsilon_{rs}^{0})\frac{\partial V_k}{\partial y_l}dy$ for any $V_k = V_k(y)$ with Y-periodicity. (8a)

or $\int_Y E_{klij}^{\varepsilon}(x,\varepsilon_{rs}^{o})\Delta\varepsilon_{ij}^{1}\delta\varepsilon_{kl}dy = \int_Y E_{klij}^{\varepsilon}(x,\varepsilon_{rs}^{o})\Delta\varepsilon_{ij}^{o}\delta\varepsilon_{kl}dy$ (8b)

Then, the micro-stress-strain relation is given by: $\Delta\sigma_{ij}(x,y;t) \approx \Delta\sigma_{ij}^{1} = E_{klij}^{\varepsilon}(x,\varepsilon_{rs}^{0})[\Delta\varepsilon_{ij}^{o} + \Delta\varepsilon_{ij}^{1}]$ (9)

$$\Delta\varepsilon_{ij}^{o} = \frac{1}{2}\left(\frac{\partial \Delta u_k^{o}}{\partial x_l} + \frac{\partial \Delta u_l^{0}}{\partial x_k}\right) \qquad \Delta\varepsilon_{ij}^{1} = \frac{1}{2}\left(\frac{\partial \Delta u_k^{1}}{\partial y_l} + \frac{\partial \Delta u_l^{1}}{\partial y_k}\right)$$

Above gives an equivalent constitutive relation, that is , strain ε_{ij}^{o} vs $\sigma_{ij}^{h} =< \sigma_{ij}(x,y) >$ is easily determined if the microstructure and all micro-constitutive laws of each constituent are known in a unit cell. The $\sigma_{ij}^{h} \sim \varepsilon_{ij}^{o}$ is consistent to that tested in Lab. What is the difference from usual one is that the interaction of microstructures is emphasized in this proposal. A program CELL.FORT is implemented to find out the homogenized constitutive law (eq.(4)), whose nonlinear equations are solved by Newton-Raphson Method or incremental method. Fig.1 shows the relationship between local problem to find $[D]^h$ out and global problem to find $\{\varepsilon^o\}$ or $\{\sigma^o\}$ out. This requires iteration techniques between local problem and global one for nonlinear constitutive laws. CELL.FORT is called one time at one step to form $[D]^h$ at each Gaussian point of every global element. Following calculation is carried out under ε^o -path given. +++Let $V_k = W_k^{ij}$ in Eq.(8a) and substitute it into usual coefficients Eq.(5) can be obtained.#

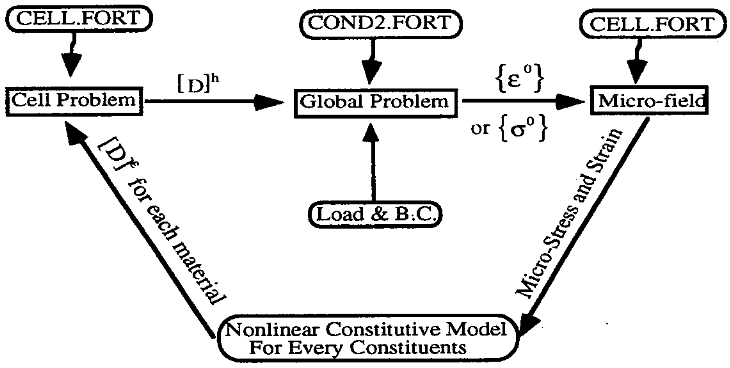

Fig.1 Numerical Scheme

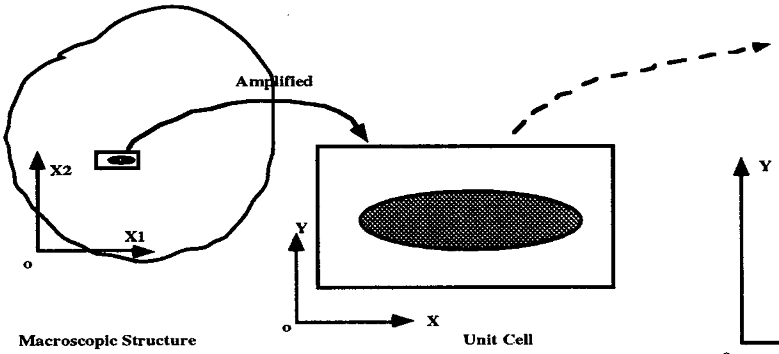

Fig.2(a) The Periodic Microstructures

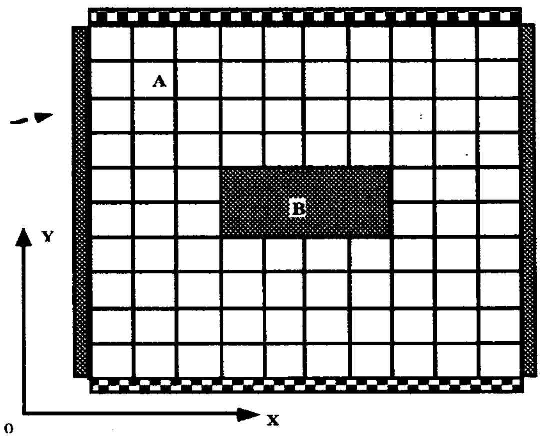

Fig.2(b) Mesh of a Unit Cell

4. Nonlinear constitutive laws of each constituent

Duncan & Chang (1970,1972; 1980) proposed a nonlinear model as followings. There are a lot of reference parameters for this model (T.D. Stark, 1994)[21]:

Loading

$$E_t = K \bullet P_a \left(\frac{\sigma_3 + \sigma_d}{P_a}\right)^n \left[1 - \frac{R_f (1-\sin\phi)(\sigma_1 - \sigma_3)}{2C \bullet \cos\phi + 2\sigma_3 \sin\phi}\right]^2 \qquad \mu_t = \frac{G - F\lg\left(\frac{\sigma_3}{P_a}\right)}{(1 - D\varepsilon_1)^2}$$

$$\varepsilon_1 = \frac{(\sigma_1 - \sigma_3)}{K \bullet P_a \left(\frac{\sigma_3 + \sigma_d}{P_a}\right)^n \left[1 - \frac{R_f (1-\sin\phi)(\sigma_1 - \sigma_3)}{2C \bullet \cos\phi + 2\sigma_3 \sin\phi}\right]} \qquad \text{(10a,b,c)}$$

Unloading or Reloading

$$E_{ur} = K_{ur} \bullet P_a \left(\frac{\sigma_3}{P_a}\right)^{n_0} \qquad \mu = 0.3 \qquad \text{(11a,b)}$$

R_f is called failure ratio; C,ϕ are parameters of material strength, and E_t, μ_t, E_{ur} are elastic modulus, of loading, Poisson ratio and elastic modulus of unloading/reloading, respectively. σ_1,σ_3 are maximum principal stress and confining stress. ε_1 is the axial strain in conventional triaxial experiment. P_a is atmospheric pressure. K, K_{ur}, σ_d, n, n_0, G, F, D, C, ϕ are all parameters of the Duncan-Chang model.

$\phi = 30^\circ$, $C = 0.2$, $R_f = 0.716$, $\sigma_d = 8.0$, $K = 20.2$, $K_{ur} = 347.1$
$n = 1.094$, $n_o = 0.83$, $G = 0.31$, $F = 0.04$, $D = 2.1$ for one compacted soil from $\sigma_3 = 10kpa$.

Loading function proposed by Duncan (1978) $f_l = \dfrac{\sigma_1 - \sigma_3}{(\sigma_1 - \sigma_3)_f}(\sigma_3)^{0.25}$

Loading Criterion is: $f_l \geq (f_l)_{max}$:Loading ; $f_l < 0.75(f_l)_{max}$:Completely unloading/reloading;

$$E = E_t + (E_{ur} - E_t)\frac{1 - f_l/(f_l)_{max}}{1 - 0.75}$$:Otherwise

5. Simple Case Studies

See Fig.2(b). A is Duncan-Chang material and B is linear elasticity. Four cases are studied: Hard and Soft Cores + Strain Paths: K_o-Case and Isotropic Compression. Hard core: $E = 4000kgf/cm2$ $\mu = 0.3$ Soft core: $E = 200kgf/cm2$ $\mu = 0.3$. Homogeneous material refers to there is only material A no B.

5.1 Comparison between incremental method (no iteration) and Newton-Raphson method (iteration)

Discretized eq.(8) can be expressed as: $\mathbf{K}(\mathbf{a})\mathbf{a} = \mathbf{F}(\mathbf{a})$ (12)

a is characteristic function vector or fluctuation displacement u^1 on spatial variable $\mathbf{y}$. $K(a) = LDL'$ is used to save computation time. There are two methods to solve nonlinear equation (12): Incremental method (No Iteration) and Newton-Raphson iterative technique (Iteration). Fig.3 is the effect of different steps in incremental method. The step has a little effect on the convergence (see Fig.3(a)), but when step is small enough, results are almost the same, see Fig.3(b). Fig.4 shows that iterative technique and incremental method have a little difference, it means that under usual condition incremental method is enough to reach the required precision. Fig.5 is equivalent stress-strain of x direction. There is no vital difference for two methods. Furthermore, it may be surprising that the Young's moduli decrease first and increase then for Duncan-Change model under Ko-case loading, see Fig.6(a). This means that Duncan-Chang Model is not proper to Ko-case or proportional loading. The stress-path for lateral constraint case is nearly straight line (see Fig.6(b)).

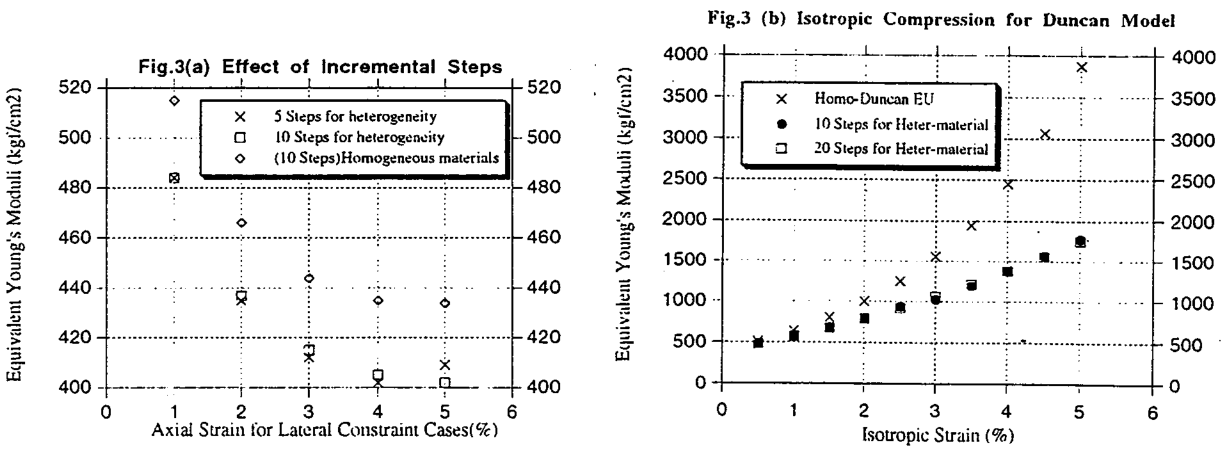

Fig.3 Effect of Step Length for Incremental Method

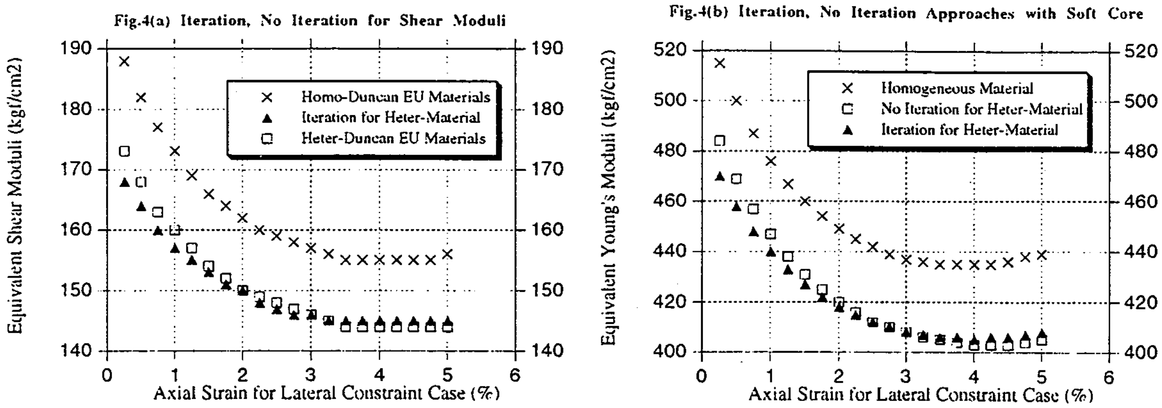

Fig.4 Comparison between Incremental Method and Newton-Raphson Method

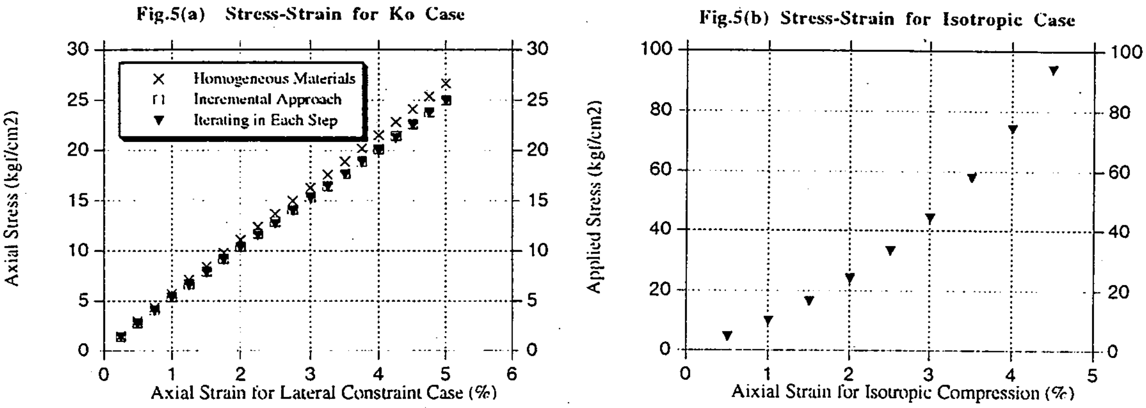

Fig.5 Equivalent Stress (in X direction) -Strain Curves

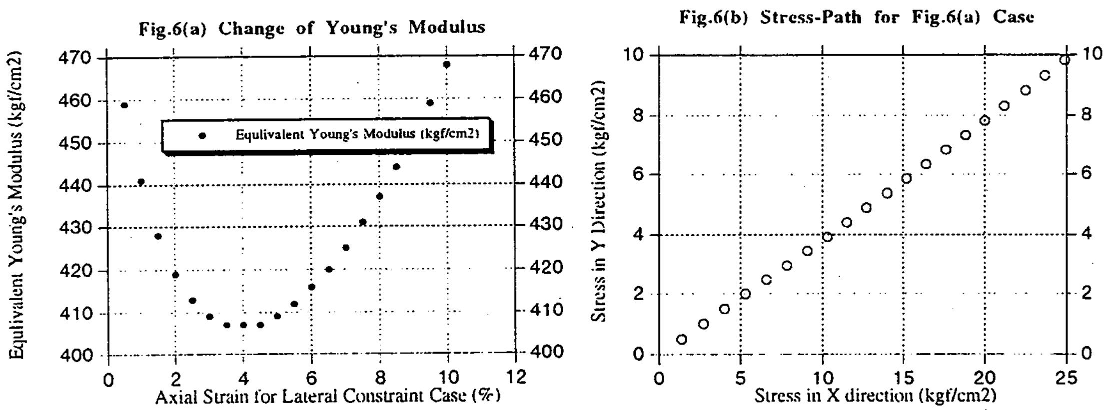

Fig.6 The Evolution of Young's Moduli and Stress Path

5.2 Initial anisotropy

Fig.7 shows the effect of initial anisotropy for hard core and soft core. It shows that initial anisotropy will vanish with deformation precedes. If the two materials are similar each other, the initial anisotropy vanish quickly (Fig.7(a,b)). But the difference between adjoining materials is too large to vanish completely or vanish slowly (Fig.7(c)). Above behaviors can be explained from the evolution of stress level contour during deformation.

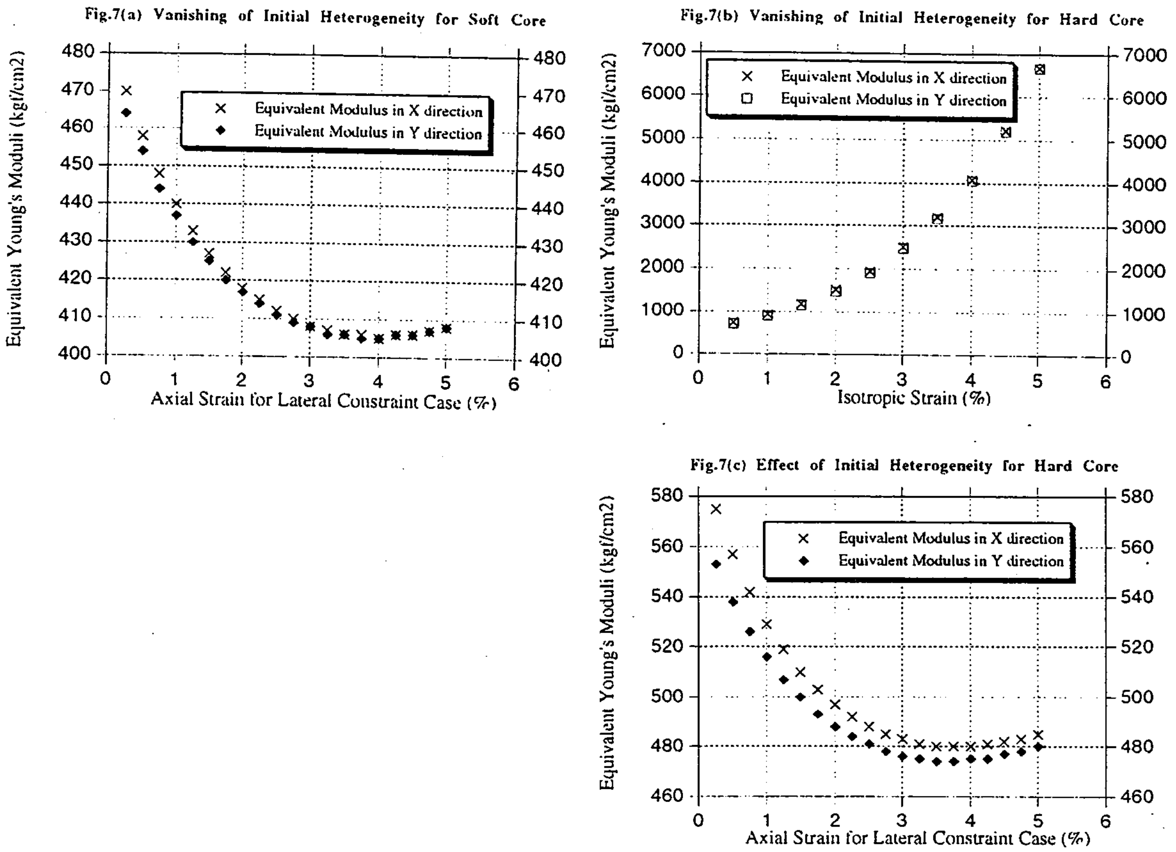

Fig.7 Vanishing of Initial Anisotropy during Deformation

5.3 Microstructural effect on mechanical properties

If the volume fraction of inclusions is kept 25% (Scheme 3 is 26%) and the inclusions have different patterns in a unit cell (Fig.8). The evolution of moduli is shown in Fig.9. It shows that the uniform distribution has highest deformation moduli. But the concentrated distribution is the weakest one.

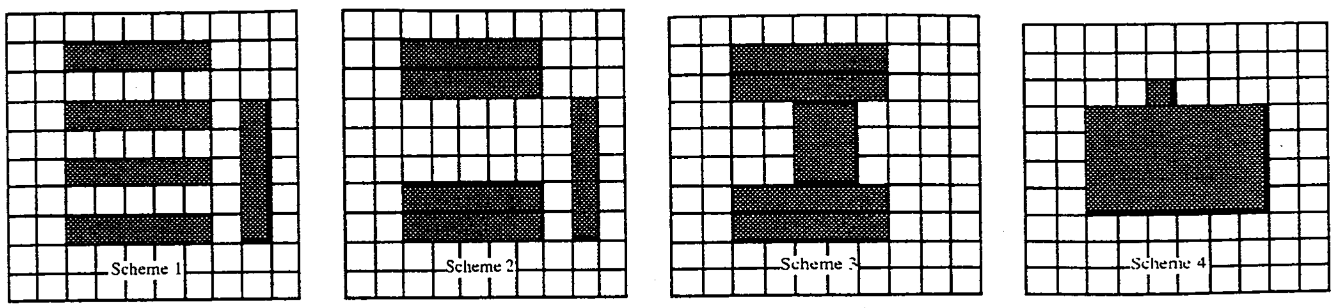

Fig.8 Four Patterns of Microstructures

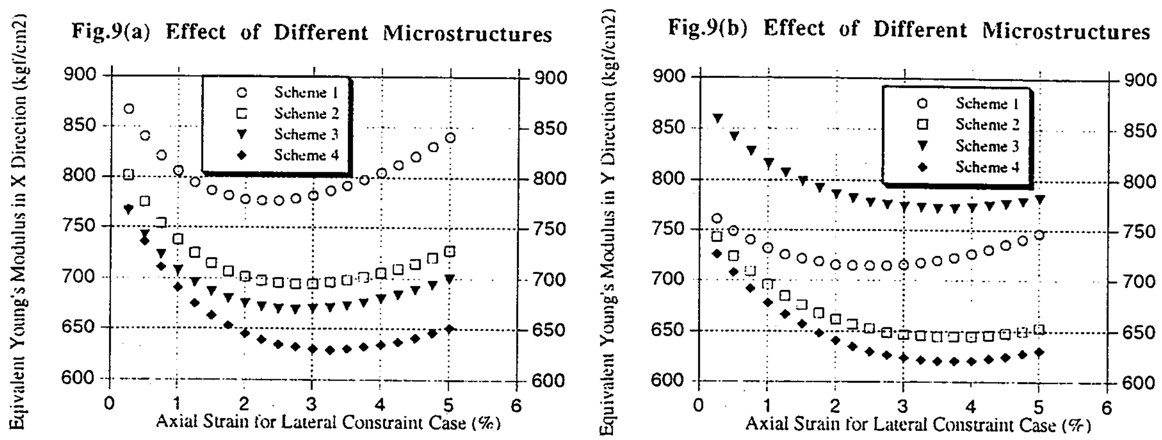

Fig.9 The Evolution of Young's Moduli during Loading

5.4 Effect of volume fraction

It should be noticed that the failure problem is a little different from deformation problem because the controlling factors are different. The failure problem will be discussed in another paper.

5.5 The induced anisotropic effect

The induced anisotropy is induced by the residual stress, which is a self- equilibrated stress in a unit cell but has vital effect on coming mechanical behaviors because of nonlinearity of materials.

Some detailed discussion will be found in [25].

6. Conclusions

(1) The homogenized constitutive laws are not only dependent on the mechanical behaviors of all constituents, but also dependent on the microstructures. The distribution of micro-stress or micro-strain field has vital effect on the macro-behaviors or coming mechanical behaviors and the nonlinearity of materials intensifies this effect.

(2) Nonlinear materials have self-cured capability because the nonlinearity can adjust the micro-stress field to reduce the difference in micro-stress. This is why the initial anisotropy may vanish gradually with the deformation.

Above studies are not enough to conduct such conclusions, further investigation is being made for various materials and micro-constitutive laws in Ichikawa's Laboratory of Nagoya University.

The assistance of Mr. Tsuji and Mr. Ogasawana, Master students in Ichikawa's Laboratory of Nagoya University, are gratefully acknowledged to provide a figure-out program.

7. References

[1] Toshio Mura(1987), *Micromechanics of defects in solids*, Second, revised edition, Martinus Nijhoff Publishers

[2] H.Horii and S. Nemat-Nasser(1983), Overall moduli of solids with microcracks: Load-induced anisotropy, J of Mechanics, Physics and Solids, 31(2):155-171

[3] M.Hori and T. Miura(1992), On overall elasticity of solid damaged by periodically distributed cracks, Japanese J of Structural Engineering, 38A:351-359

[4] E. Sanchez-Palencia(1980), Non-homogeneous media and vibration theory, Lecture Notes in Physics, Vol.127, Springer Berlin

[5] A.E. Saez, J.C. Perfetti and Isak Rusinek(1991), Prediction of effective diffusivities in porous media using spatially periodic models, Transport in Porous Media, 6:143-157

[6] Stefan Jansson(1992), Homogenized nonlinear constitutive properties and local stress concentrations

for composites with periodic internal structure, Int. J of Solids and Structures, 29(17):2181-2200
[7] F.Devries, H.Dumontet, G. Duvaut and F. Lene(1989), Homogenization and damage for composite structures, Int. J of Numerical Methods in Engineering, 27:285-298
[8] G.Allaire and F. Murat(1993), Homogenization of the Neumann problem with nonisolated holes, Asymptotic Analysis, 7:81-95
[9] Weinan E(1992), Homogenization of linear and nonlinear transport equations, Communications on Pure and Applied Mathematics, 45:301-326
[10] G. Allaire(1992), Homogenization and two-scale convergence, SIAM J on Mathematical Analysis, 23(6):1482-1518
[11] L. Boccardo and T. Del Vecchio(1991), Homogenization of strongly nonlinear equations with gradient dependent lower order nonlinearity, Asymptotic Analysis, 5:75-90
[12] H.Fan, L.M.Keer and T. Mura(1992), Inhomogeniety problem revisited via the modulus perturbation approach, Int. J of Solids Structures, 29(20):2583-2594
[13] J.L. Auriault, D. Bouvard, C. Dellis and M. Lafer(1992), Modelling of hot compaction of metal powder by homogenization, Mechanics of Materials, 13:247-255
[14] G.dal Maso and A. Defranceschi(1990), Correctors for the homogenization of monotone operators, Differential and Integral Equations, 3(6):1151-1166
[15] G.P. Pansenko(1990), Multicomponent homogenization of processes in strongly nonhomogeneous structures, Mathematics of USSR Sbornik, 69(1):143-153
[16] F.Devries, H.Dumontet, G.Duvaut and F. Lene(1989), Homogenization and damage for composite structures, Int. J for Numerical Methods in Engineering, 27:285-298
[17]F.Santosa and W.W.Symes(1989), A model for a composite with anisotropic dissipation by homogenization, Int. J of Solids Structures, 25(4):381-392
18] M.Ortiz(1987), A method of homogenization of elastic media, Int. J of Engineering Science, 25(7):923-934
[19] M.Arminjon(1991), Limit distributions of the states and homogenization in random media, Acta Mechanica, 88:27-59
[20] S.C. Mandal and M.L.Ghosh(1994), Interaction of elastic waves with a periodic array of coplanar Giffith cracks in an orthotropic elastic medium, Int. J of Engineering Science, 32(1):167-178
[21] T.D. Stark, R. M. Ebeling and J.J. Vettel(1994), Hyperbolic stress-strain parameters for silts, ASCE, J of Geotechnical Engineering, 120(2):420-441
[22] S.A. Paipetis, D. Polyzos and M. Valavanidis(1993), Constitutive relations of periodic laminated composites with anisotropic dissipation, Archive of Applied Mechanics, 64:32-43
[23] Jian-Guo Wang, Y. Ichikawa and M. Wani(1994), Studies on effect of various micro-structures on elastic moduli by homogenization method, To appear in Proceeding of 49th National Conference of JSCE. Full paper is in Ichikawa's Laboratory of Nagoya University
[24] Jian-Guo Wang and Y. Ichikawa(1994), One dimensional nonlinear model of sand densification by homogenization method, Proceeding of Chubu Regional Civil Engineering Meeting. Full paper is in Ichikawa's Laboratory of Nagoya University
[25] Jian-Guo Wang, Y. Ichikawa and K. Tsuji(1994), The evolution of micro-stress and macro-mechanical behaviors of nonlinear geomaterials, To be submitted to 38th Jointed Conference of Material of Japan
[26] Instructions on CELL.FORT(2-DIM) and COND2.FORT(2-DIM), Ichikawa's Laboratory, Dept. of Geotechnical and Environmental Engineering, Nagoya University, February 1994